Spaces of primitive elements in dual modules over Steenrod algebra 2

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I present a way to generate all primitive elements $PB(n)$ in $B(n) = (A(n-1)/A(n))^*$ modules over A^* dual Steenrod algebra, where $A(n)$ are annihilator modules over Steenrod algebra A. This work began in [7]. For useful notions see also [1, 2, 3, 4] and references summarized in [8, 5, 6, 7]. The filtration described in [6] Theorem 1 property 2 and 3 yields $PB(n) = \bigcup_t PB(n)_t$ and $PB(n)_t = \bigoplus_s PB(n)_t^s$, where s is the number of τ operations and t is the biggest index of such operations. From Theorem 1 [7] property 5 and 1 it is known $dimPB(n)_t^{s,deg} \leq 1$ and the following diagram is exact

$$0 \to PC(n)_{k-1} \xrightarrow{\iota_k} PC(n)_k \xrightarrow{\lambda_k} PC(n-1)_{k-1}$$

For given $\alpha \in PB(n-1)_{t-1}$ how to find a primitive $\alpha' \in PB(n)_t$ such that $\pi_t(\alpha') = \alpha$? Properties 2 and 6 state for even n that $PB(n)_{-1} = PB(n)_0 = \langle \xi_1^{n/2} \rangle$; and for odd n: $PB(n)_0 = \langle \xi_1^{\frac{n-1}{2}} \tau_0 \rangle$ and for $s \geq 1$ that $\alpha\tau_0$ is also a primitive. And we can generate new primitives taking products of primitives. Do all primitives in $PB(n)_k^1 \backslash PB(n)_0^1$ also have form $\alpha\tau_0$? So $\alpha\tau_1 \in B(n)$ yields coproduct $\phi^*(\alpha\tau_1) = \phi^*(\alpha)(\xi_1 \otimes \tau_0 + 1 \otimes \tau_1)$ and hence $\alpha\tau_1$ is primitive if and only if $\alpha = \alpha'\tau_0 \in PB(n-1)$. If $\alpha \in PB(n)_{-1}$ then $\alpha\tau_1 = \xi_1^{\frac{n}{2}}\tau_1$ is not primitive. But for example product of not primitive $\alpha = \xi_1^{\frac{n-1}{2}} + \tau_0\xi_2^{\frac{n-1}{2}} \in B(n)_0^1$ with primitive τ_0 is primitive. The primitivity condition in $B(n)$ leads to the following inductive definition of transformations R_i generating primitives, preserving primitivity.

Означення 1.

$$R_k(\alpha) = \xi^{\frac{p^{k-1}-1}{p-1}} \tau_k \alpha - \sum_{i=1}^{k-1} \xi^{\frac{p^{k-1}-p^i}{p-1}} \xi_{k+1-i}^{p^{i-1}} R_i(\alpha)$$

for $k > 1$ and $R_0(\alpha) = \alpha\tau_0$, $R_1(\alpha) = \alpha\tau_1$

These maps have the following properties.

Теорема 2. *(1)* $\forall i, k \in N$, $\forall \alpha \in B$: $R_i(\alpha\tau_k) = -R_i(\alpha)\tau_k$

(2) $\forall i, k \in N$, $\forall \alpha \in B$: $R_i(\alpha\xi_k) = R_i(\alpha)\xi_k$

(3) $\forall i, j \in N$, $\forall \alpha \in B$: $R_iR_j(\alpha) = -R_jR_i(\alpha)$

(4) $\forall \alpha \in PB(n) \cap ImR_0$: $R_i(\alpha) \in PB(n + 1 + 2\frac{p^{i-1}-1}{p-1})$

Зауваження 3. From the definition 1: $R_i(\alpha) = \alpha R_i(1)$. Therefore by induction $R_{i_1}R_{i_2}\cdots R_{i_k}(\alpha) = \alpha R_{i_1}R_{i_2}\cdots R_{i_k}(1)$. And for example $R_2(1)$, $R_3(1)$ e.t.c. are primitives in $B(n)^1$.

Therefore all primitives have form $\alpha\tau_0$ except $PB(n)_k^1 \backslash PB(n)_0^1$. Induction arguments based on Theorem 1 [7] lead to the general form of primitive elements.

Означення 4. $\alpha_{i_1,i_2,\ldots i_k} = \xi_1^l \tau_{i_1}\tau_{i_2}\ldots\tau_{i_k} + \beta$ is a primitive in $PB(n)_{i_k}^k$ associated with $(i_1, i_2, \ldots i_t)$, $i_k > i_{k-1} > \ldots > i_1 = 0$ if it has projection on $J(n)^{k,deg} = B(n)^{k,deg}/(I \cap B(n)^{k,deg})$ equal $a\xi_1^l\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}$, $l = \frac{n-k}{2}$, $a \in Z/p$.

Наслідок 5. *There exists the primitive* $\alpha_{i_1,i_2,\ldots i_k}$ *associated with* $(i_1, i_2, \ldots i_k)$, $i_k > i_{k-1} > \ldots > i_1 = 0$ *and it is satisfied* $\alpha_{i_1,i_2,\ldots i_k}\xi_1^l = R_{i_1}R_{i_2}\cdots R_{i_k}(1)$.

Зауваження 6. Corollary 5 also presents a way to calculate all associated primitives.

The following theorem is a result of construction of all primitive elements in $B(n)$.

Теорема 7. *All* $PB(n)^{s,deg}$ *in* $PB(n) = \cup_k PB(n)_k$ *where* $PB(n)_k = \oplus PB(n)^s_k$ *are zero or one dimensional spaces.* $PB(n)^{s,deg}$ *has dimension one if and only if there is a sequence* $(i_1, i_2, \dots i_t)$, $i_s > i_{s-1} > \dots > i_1 = 0$ *with conditions*

(1) $n - s$ *is even,*

(2) degree of $PB(n)^{s,deg}$ *is* $deg = (p-1)(n-s) + \sum_{j=1}^{s} dim(\tau_{i_j})$,

(3) $\frac{n-s}{2} \geq l$, *where* l *is calculated below:*

(4) $l = \sum_{j=2}^{s} \frac{p^{i_j - 1} - 1}{p-1} - \sum_{j=2}^{s-1} \frac{p^{i_j} - 1}{p-1}$,

When $\frac{n-s}{2} = l$

$$PB(n)^{s,deg} = \langle \alpha_{i_1, i_2, \dots i_s} \rangle$$

When $\frac{n-s}{2} > l$

$$PB(n)^{s,deg} = \langle \xi_1^{\frac{n-s}{2} - l} \alpha_{i_1, i_2, \dots i_s} \rangle$$

where $\alpha_{i_1, i_2, \dots i_s} = \xi_1^l \tau_{i_1} \tau_{i_2} \cdots \tau_{i_s} + \beta$ *is the primitive in* $B(n)^s_{i_s}$ *associated with the sequence* $(i_1, i_2, \dots i_t)$, $i_s > i_{s-1} > \dots > i_1 = 0$ *with conditions 1,2,4 mentioned above and* $\frac{n-s}{2} = l$.

Knowledge of primitive elements on $B(n) = (A(n-1)/A(n))^*$ make a feasible to find all indecomposable elements of $(A(n-1)/A(n)$ [8, sec 4] .

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